Journal of Engineering Mathematics **26**: 131–152, 1992.
H.K. Kuiken and S.W. Rienstra (eds), Problems in Applied, Industrial and Engineering Mathematics.

The Stewartson layer of a rotating disk of finite radius

A.I. van de VOOREN
Department of Mathematics, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands

Abstract. It is shown that if a disk of finite radius and the surrounding medium rotate coaxially with slightly different angular velocities, an axial layer in the form of a cylindrical shell exists at the edge of the disk. This shell of thickness $O(E^{1/3})$ has length $O(E^{-1})$ in axial direction, where E is the Ekman number. Its most characteristic element is the axial velocity of $O(E^{1/6})$ which is larger than everywhere else in the field. We calculate the velocity components and the pressure in this layer.

1. Introduction

In 1957 Stewartson published a paper [1] in which he considered the shear layers existing between two coaxial rotating planes of which the center disks rotate with a slightly different angular velocity. He found that if the deviations of the angular velocities of the disks from that of the planes are equal but opposite, a shear layer of thickness $E^{1/3}$ exists, while if the deviations are equal in the same sense an additional layer of thickness $E^{1/4}$ appears. This last layer is necessary in order to fit the azimuthal velocity of the inner region to that of the outer region. In the following such shear layers will be denoted as Stewartson layers. E is the Ekman number.

Greenspan gave in his monograph [2] a clear account of these Stewartson layers while Moore and Saffman [3] presented an analysis of different possibilities for a variety of situations. Hide and Titman [4] performed an experimental investigation on a rotating disk of finite radius placed in a cylindrical tank which itself is rotating with a slightly different angular velocity. They showed the existence of the Stewartson layer.

In the present investigation a disk of radius a rotates with an angular velocity Ω in an unbounded medium which itself rotates coaxially with angular velocity $(1-\varepsilon)\Omega$. Our problem will be linearized in the small Rossby number ε. The configuration is clarified in Fig. 1, where the various regions of the flow field are indicated.

The rotation of the medium can physically be realized by thinking of a cylindrical tank rotating with an angular velocity $(1-\varepsilon)\Omega$. Top and bottom of the tank must have a distance to the disk larger than $O(E^{-1})$ in order that the Stewartson layer of the disk is not influenced by top and bottom of the tank (see also [3]).

It will be shown that for our configuration there exists a Stewartson layer of thickness $E^{1/3}$. There is no layer of thickness $E^{1/4}$ since at both sides of the Stewartson layer (regions III and IV) the angular velocities are equal, viz. $(1-\varepsilon)\Omega$. Velocity and pressure distributions are dependent upon a similarity parameter $\tau = z/r_1^3$, where r_1 is the stretched radial coordinate in the Stewartson layer. In this way expressions for velocity and pressure distributions are derived in the form of integrals, which have been evaluated by Romberg integration.

At the point $z=0$, $r_1=0$, which is the point, where the Stewartson layer is joined to the Ekman layer [3], there exists a logarithmic singularity in the pressure. This singularity is responsible for the deflection of the boundary layer flow to the axial flow in the Stewartson layer.

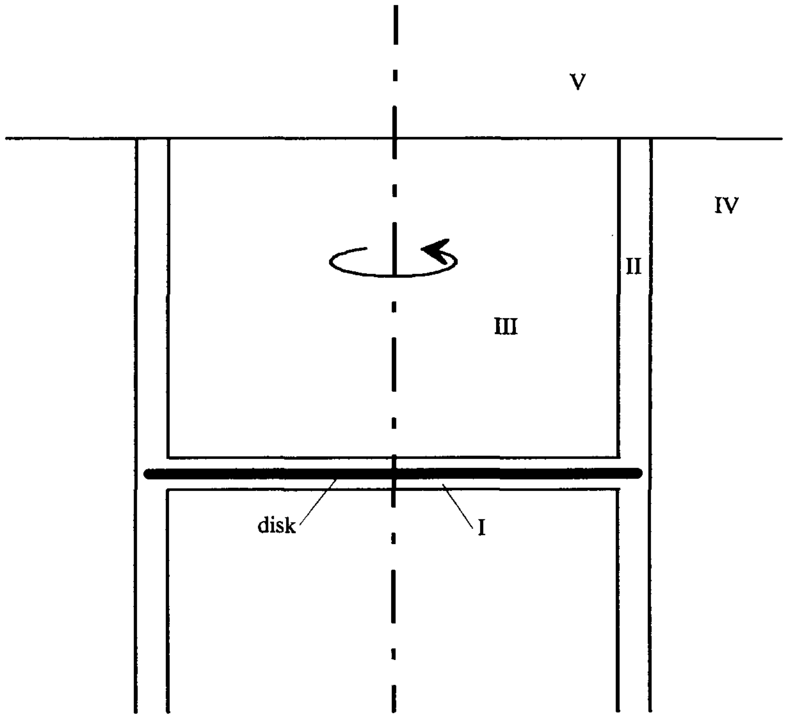

Fig. 1. The configuration. I = Ekman layer, II = Stewartson layer, III = inner region, IV = outer region, V = upper region.

The Stewartson layer II has small effects upon the region III and IV in the same way as the Ekman layer I induces an axial velocity in region III.

2. General equations

For an axially symmetric configuration the dimensionless equations of motion are in an inertial system of reference:

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}=-\frac{\partial p}{\partial r}+E\left\{\frac{\partial^2 u}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{u}{r}\right)+\frac{\partial^2 u}{\partial z^2}\right\},$$
$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=E\left\{\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v}{r}\right)+\frac{\partial^2 v}{\partial z^2}\right\},\qquad(2.1)$$
$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+E\left\{\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right\},$$

while the equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{\partial w}{\partial z}=0\,.\qquad(2.2)$$

u, v and w are the radial, azimuthal and axial velocities, respectively. p is the pressure, $E=\nu/\Omega a^2$ the Ekman number with ν the kinematical viscosity coefficient. Lengths have been made dimensionless with a, velocities with Ωa and the pressure with $\rho\Omega^2 a^2$ where ρ is the fluid density. In order to satisfy the equation of continuity a stream function ψ is introduced by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{2.3}$$

The boundary conditions are

$$\left.\begin{array}{l} \text{at the disk} \\ z = 0, \quad r < 1: \quad u = 0, \quad v = r, \quad w = 0, \\ \text{at infinity} \\ \sqrt{z^2 + r^2} \to \infty: \quad u \to 0, \quad v \to (1-\varepsilon)r, \quad w \to 0, \quad p \to \frac{1}{2}(1-\varepsilon)^2 r^2. \end{array}\right\} \tag{2.4}$$

3. The Ekman layer

Since the Rossby number is infinitely small, all deviations from the original flow will be proportional to ε. Introducing the boundary layer coordinate $\tilde{z} = E^{-1/2} z$, we may write

$$\begin{aligned} &\psi = \frac{1}{2}\varepsilon E^{1/2} r^2 h(\tilde{z}), \qquad u = \frac{1}{2}\varepsilon r h'(\tilde{z}), \qquad v = r - \varepsilon r g(\tilde{z}), \\ &w = -\varepsilon E^{1/2} h(\tilde{z}), \qquad p = \frac{1}{2}(1 - 2\varepsilon) r^2. \end{aligned} \tag{3.1}$$

Substitution in the equation (2.1) leads to

$$-h''' + 4g = 4, \qquad h' + g'' = 0, \tag{3.2}$$

while the third equation (2.1) shows that $\partial p / \partial \tilde{z} = O(E)$. The linearization in (3.2) is valid for $|\varepsilon| < O(1)$. Boundary conditions for (3.2) are

$$\begin{aligned} &\tilde{z} = 0: \quad h = 0, \quad h' = 0, \quad g = 0, \\ &\tilde{z} \to \infty: \quad h' \to 0, \quad g \to 1. \end{aligned} \tag{3.3}$$

By elementary methods the solution of this system is obtained as

$$g = 1 - e^{-\tilde{z}} \cos \tilde{z}, \qquad h = 1 - e^{-\tilde{z}}(\sin \tilde{z} + \cos \tilde{z}). \tag{3.4}$$

We now investigate whether the boundary layer exists also for $r > 1$ as is the case for a rotating disk in a fluid at rest, see van de Vooren and Botta [5]. The boundary conditions (3.3) are replaced outside the disk by

$$\begin{aligned} &\tilde{z} = 0: \quad h = 0, \quad h'' = 0, \quad g' = 0, \\ &\tilde{z} \to \infty: \quad h' \to 0, \quad g \to 1. \end{aligned}$$

By the same elementary methods as used earlier, it is found that the only solution is $g = 1$, $h = 0$. This means that outside the disk we can only have the original flow with $v = (1 - \varepsilon)r$.

Hence, at $r = 1$ the Ekman layer suddenly ends, which means that there occur large changes in radial direction and this gives rise to a Stewartson layer.

Finally, we calculate the torque acting on the disk. The tangential shear stress at the disk is

$$\tau_0 = \rho\Omega^2 a^2 E^{1/2} \frac{\partial v}{\partial \tilde{z}}$$

and the torque is $M = 2\pi a^3 \int_0^1 \tau_0 r^2 \, dr$.

With $\partial v/\partial z = -\varepsilon r g'(0)$ and $g'(0) = 1$, we find

$$M = -\frac{1}{2} \varepsilon\pi\rho\Omega^2 a^5 E^{1/2} . \tag{3.5}$$

A negative value of M has a decelerating effect on the disk.

4. Equations in the Stewartson layer

The scaling of the various quantities in the Stewartson layer can be taken most easily from Greenspan [2], pp. 98 and 99. To comply with the rapid changes in radial direction, a stretched coordinate r_1 is introduced by

$$r = 1 + E^{1/3} r_1 . \tag{4.1}$$

r_1 and z are the independent variables in the Stewartson layer. The dependent variables are expanded as follows

$$\begin{aligned}
\psi &= \varepsilon E^{1/2}\psi_1 + \varepsilon E^{5/6}\psi_2 + \cdots \\
u &= \varepsilon E^{1/2}u_1 + \varepsilon E^{5/6}u_2 + \cdots \\
v &= (1-\varepsilon)r + \varepsilon E^{1/6}v_1 + \varepsilon E^{1/2}v_2 + \cdots \\
w &= \varepsilon E^{1/6}w_1 + \varepsilon E^{1/2}w_2 + \cdots \\
p &= \frac{1}{2}(1-\varepsilon)^2 r^2 + \varepsilon E^{1/2}p_1 + \varepsilon E^{5/6}p_2 + \cdots .
\end{aligned} \tag{4.2}$$

This gives an axial flux $O(E^{1/2})$ which is the deflected radial flux of $O(E^{1/2})$ existing in the Ekman layer. The second terms in the expansion to E are a factor $E^{1/3}$ smaller than the first terms. This is in agreement with (4.1) and (3.1) and, moreover, the second term in the expansion of w is required to match the term $w = -\varepsilon E^{1/2}$, present in the inner region as follows from (3.1) and (3.4).

Substitution of (4.2) into the equations (2.1) and (2.2) leads to the following set of equations for the first approximation

$$2v_1 = \frac{\partial p_1}{\partial r_1} , \qquad 2u_1 = \frac{\partial^2 v_1}{\partial r_1^2} , \qquad \frac{\partial p_1}{\partial z} = \frac{\partial^2 w_1}{\partial r_1^2} , \qquad \frac{\partial u_1}{\partial r_1} + \frac{\partial w_1}{\partial z} = 0 . \tag{4.3}$$

The terms of the third equation come from terms in (2.1) which are $O(\varepsilon E^{1/2})$, while the neglected non-linear terms in this equation are $O(\varepsilon^2 E^{1/3})$. Hence, it is required that $|\varepsilon| < O(E^{1/6})$.

For the second approximation the following set is obtained

$$2v_2 = \frac{\partial p_2}{\partial r_1}, \qquad 2u_2 = \frac{\partial^2 v_2}{\partial r_1^2} + \frac{\partial v_1}{\partial r_1}, \qquad \frac{\partial p_2}{\partial z} = \frac{\partial^2 w_2}{\partial r_1^2} + \frac{\partial w_1}{\partial r_1}, \qquad \frac{\partial u_2}{\partial r_1} + \frac{\partial w_2}{\partial z} + u_1 = 0. \tag{4.4}$$

This approximation is only valid if $|\varepsilon| < O(E^{1/2})$.

From equations (2.3) and (4.1) we find

$$u_1 = \frac{\partial \psi_1}{\partial z}, \qquad u_2 = \frac{\partial \psi_2}{\partial z} - r_1 \frac{\partial \psi_1}{\partial z}, \qquad w_1 = -\frac{\partial \psi_1}{\partial r_1}, \qquad w_2 = -\frac{\partial \psi_2}{\partial r_1} + r_1 \frac{\partial \psi_1}{\partial r_1}. \tag{4.5}$$

Elimination of all variables except ψ_1 and ψ_2 yields as fundamental equations

$$\frac{\partial^6 \psi_1}{\partial r_1^6} + 4\frac{\partial^2 \psi_1}{\partial z^2} = 0, \qquad \frac{\partial^6 \psi_2}{\partial r_1^6} + 4\frac{\partial^2 \psi_2}{\partial z^2} = 3\frac{\partial^5 \psi_1}{\partial r_1^5}. \tag{4.6}$$

The Ekman layer which has thickness $O(E^{1/2})$ is reduced in the z-coordinate to $z = 0$. Hence $z = 0$ must correspond to the outer edge of the Ekman layer. Moreover, the equations in the Ekman layer are only modified by a stretched coordinate

$$r = 1 + E^{1/2}\tilde{r},$$

since only then the second derivatives to r become of the same order as second derivatives to z. Thus, in the r_1-coordinate this modification occurs at $r_1 = 0$. For $r_1 < 0$ we have at the outer edge of the Ekman layer using (3.1), (3.4) and (4.1)

$$z = 0: \quad \psi = \frac{1}{2}\epsilon E^{1/2} + \varepsilon E^{5/6} r_1 + \frac{1}{2}\varepsilon E^{7/6} r_1^2.$$

Thus $\psi_1 = \frac{1}{2}$, $\psi_2 = r_1$, $w_1 = 0$, $w_2 = -1$.

For $r_1 > 0$ where there is no Ekman layer we have

$$\psi_1 = 0, \quad \psi_2 = 0, \quad w_1 = 0, \quad w_2 = 0.$$

Hence, the boundary condition for ψ_1 is

$$z = 0: \quad \psi_1 = \frac{1}{2}\{1 - U(r_1)\} = \frac{1}{2}U(-r_1), \tag{4.7}$$

where $U(x)$ is the unit step function

$$U(x) = 1 \quad \text{for } x > 0, \qquad U(x) = 0 \quad \text{for } x < 0.$$

The other boundary conditions are

$$z \to \infty: \qquad \psi_1 \text{ is bounded},$$

$r_1 \to -\infty$: ψ_1 is bounded ,

$r_1 \to \infty$: $\psi_1 \to 0$.

Boundary conditions for ψ_2 will be given later.

5. Main solution in the Stewartson layer

The stream function ψ_1 is solved by aid of Fourier transformation

$$F_1(\omega, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_1(r_1, z)\, e^{i\omega r_1}\, dr_1 .$$

We take $\operatorname{Im} \omega < 0$ since $\psi_1 \neq 0$ for $r_1 \to -\infty$.

The transformed equation becomes

$$4 \frac{d^2 F_1}{dz^2} - \omega^6 F_1 = 0 .$$

The boundary condition for $z = 0$ is

$$F_1(\omega, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2} \{1 - U(r_1)\}\, e^{i\omega r_1}\, dr_1 = \frac{1}{2i\omega\sqrt{2\pi}} .$$

Hence,

$$F_1(\omega, z) = \frac{1}{2i\omega\sqrt{2\pi}}\, e^{-|\omega|^3 z/2} \quad \text{for } z \geq 0 .$$

Transforming back we obtain

$$\psi_1(r_1, z) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\omega r_1}}{\omega}\, e^{-|\omega|^3 z/2}\, d\omega , \tag{5.1}$$

where the path of integration has to pass below the pole $\omega = 0$. We write

$$\psi_1(r_1, z) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\omega r_1}}{\omega} (e^{-|\omega|^3 z/2} - 1)\, d\omega + \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\omega r_1}}{\omega}\, d\omega .$$

In the first integral $\omega = 0$ is no longer a singular point, so we can integrate straight along the ω-axis. The integration path of the second integral is closed by the infinitely large semi-circle in the half plane $\operatorname{Im} \omega > 0$ if $r_1 < 0$ and by the semi-circle in the half plane $\operatorname{Im} \omega < 0$ if $r_1 > 0$. By the residu theorem this gives $\frac{1}{2} U(-r_1)$ as a result for the second integral. Then

$$\psi_1(r_1, z) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{\cos \omega r_1 - i \sin \omega r_1}{\omega} (e^{-|\omega|^3 z/2} - 1)\, d\omega + \frac{1}{2} U(-r_1) .$$

Since the integrand with $\cos \omega r_1$ is odd in ω it gives no contribution and we retain

$$\psi_1(r_1, z) = \frac{1}{2\pi} \int_0^{\infty} \frac{\sin \omega r_1}{\omega} (1 - e^{-\omega^3 z/2})\, d\omega + \frac{1}{2} U(-r_1) . \tag{5.2}$$

For $z \to \infty$ the value of the integral is $\pi/2$ if $r_1 > 0$ and $-\pi/2$ if $r_1 < 0$. This means that

$$\psi_1(r_1, \infty) = \frac{1}{4} \quad \text{for all finite values of } r_1 \text{ and}$$

$$\psi_1(0, z) = \frac{1}{4} \quad \text{for all finite values of } z.$$

The integral in (5.2) is an odd function of r_1 and hence needs only to be calculated for $r_1 > 0$. Putting $\omega r_1 = y$, we can write for $r_1 > 0$

$$\begin{aligned} \psi_1(r_1, z) &= \frac{1}{2\pi} \int_0^\infty \frac{\sin y}{y} (1 - \mathrm{e}^{-y^3\tau/2})\, \mathrm{d}y \\ &= \frac{1}{4} - \frac{1}{2\pi} \int_0^\infty \frac{\sin y}{y} \mathrm{e}^{-y^3\tau/2}\, \mathrm{d}y\,. \end{aligned} \tag{5.3}$$

This formula shows that ψ_1 only depends upon the similarity parameter

$$\tau = z/r_1^3\,. \tag{5.4}$$

Moreover, for $r_1 \to 0$, $\tau \to \infty$, (5.3) gives again $\psi_1 = 1/4$, so (5.3) is valid for $r_1 \geqslant 0$. For $r_1 \leqslant 0$ we have

$$\psi_1(r_1, z) = \frac{1}{4} + \frac{1}{2\pi} \int_0^\infty \frac{\sin y}{y} \mathrm{e}^{y^3\tau/2}\, \mathrm{d}y\,.$$

For any finite value of z and r_1 varying from $-\infty$ to $+\infty$, ψ_1 varies from 1/2 to 0. Thus, the axial mass flow in the Stewartson layer is for any finite z equal to $2\pi\varepsilon E^{1/2}/2$. This is exactly equal but opposite to the axial mass flow in the inner region which is

$$-2\pi\varepsilon E^{1/2} \int_0^1 h(\infty) r\, \mathrm{d}r\,,$$

where $h(\infty) = 1$ is determined in the Ekman layer. For finite z there is no interchange between the two mass flows. That the axial velocity is constant in the inner region is due to the Taylor–Proudman theorem. However, there is a small viscosity $\nu = O(E)$ and this causes the axial velocity in the inner region to diminish at a distance $z = O(E^{-1})$, that is where the upper region V (Fig. 1) begins. The axial velocity in the Stewartson region diminishes with increasing z as a result of the widening of this layer proportional to $z^{1/3}$. In the upper region the two axial mass flows begin to annihilate each other.

Numerical values for ψ_1 are obtained by Romberg integration of the integral in (5.3). The infinite integration interval is ended when the integrand becomes smaller than 10^{-9}. Results for $\psi_1(\tau)$ are presented in Table 1 and Fig. 2.

For negative values of $\tau (r_1 < 0)$ we have

$$\psi_1(\tau) = \frac{1}{2} - \psi_1(-\tau)\,.$$

By expansion of the exponential in (5.3) we obtain in the limit $\tau \downarrow 0$

$$\psi_1(\tau) = -\frac{\tau}{2\pi} + O(\tau^3)\,.$$

Table 1. The function $\psi_1(\tau)$

τ	$\psi_1(\tau)$
0	0
0.001	−0.000 158 744
0.01	−0.002 505 524
0.03	−0.014 295 372
0.043 99	−0.016 061 169
0.05	−0.015 834 700
0.1	−0.006 200 428
0.125 785	0
0.25	0.025 184 891
0.438 175	0.05
1.339 016	0.10
5.205 917	0.15
44.867 564	0.20
∞	0.25

Negative values of $\psi_1(\tau)$ occur for $0 < \tau < 0.125\,785$. The streamlines $0 < \psi_1(\tau) < 0.5$ originate from the Ekman layer by way of the point $r_1 = 0$, $z = 0$. The main flow takes some fluid with it (negative values of ψ_1) which originates from the outer region. The minimal value of ψ_1 occurs for $\tau = 0.04399$ and this corresponds to a streamline with zero velocity. At the other side of the Stewartson layer some streamlines with $\psi_1 > 0.5$ exist, which means that some fluid is attracted from the inner region.

We can find the behaviour of ψ_1 for large values of τ by using the series expansion for sin y in (5.3). We obtain

$$I = \int_0^\infty \frac{\sin y}{y}\, \mathrm{e}^{-y^3\tau/2}\, \mathrm{d}y = \frac{2^{1/3}}{3} \int_0^\infty \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n/3}\xi^{2(n-1)/3}}{(2n+1)!}\, \mathrm{e}^{-\xi\tau}\, \mathrm{d}\xi\ ,$$

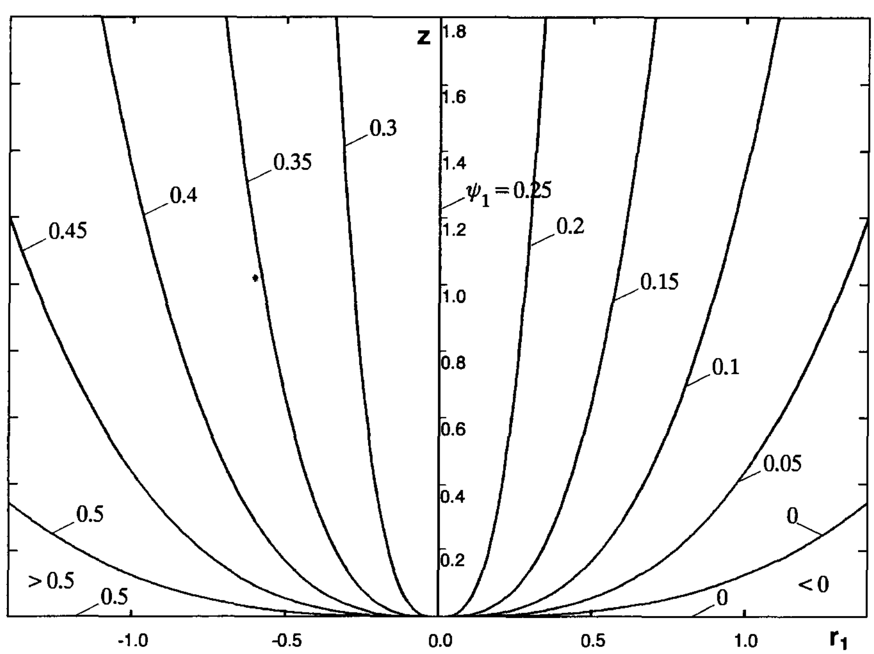

Fig. 2. Streamlines in the $r_1 - z$ plane.

where $\xi = y^3/2$. Introducing $\xi\tau = u$ as a new variable the integrals lead to Γ-functions and the result is

$$I = \frac{2^{1/3}\Gamma(\frac{1}{3})}{3\tau^{1/3}} - \frac{1}{9\tau} + \frac{2^{2/3}\Gamma(\frac{5}{3})}{180\tau^{5/3}} - \cdots$$

and

$$\psi_1 = \frac{1}{4} - \frac{1}{2\pi}\left\{\frac{2^{1/3}\Gamma(\frac{1}{3})}{3}\frac{r_1}{z^{1/3}} - \frac{r_1^3}{9z} + \frac{2^{2/3}\Gamma(\frac{5}{3})}{180}\frac{r_1^5}{z^{5/3}} - \cdots\right\}. \tag{5.5}$$

For $r_1 > 0$ the radial velocity u_1 is equal to

$$u_1 = \frac{\partial\psi_1}{\partial z} = \frac{1}{r_1^3}\frac{d\psi_1}{d\tau} = \frac{1}{4\pi r_1^3}\int_0^\infty y^2 \sin y\, e^{-y^3\tau/2}\, dy\,. \tag{5.6}$$

Furthermore, u_1 is an odd function of r_1, so $u(-r_1) = -u(r_1)$. For $\tau \downarrow 0$ which occurs if $z \downarrow 0$, $r \neq 0$ but also for z finite and $r_1 \to \infty$, we find

$$u_1 = -\frac{1}{2\pi r_1^3}\,. \tag{5.7}$$

For $\tau \to \infty$ we obtain an expression in the same way as we did for ψ_1. The result is

$$u_1 = \frac{1}{2\pi}\left\{\frac{2^{1/3}\Gamma(\frac{1}{3})}{9}\frac{r_1}{z^{4/3}} - \frac{1}{9}\frac{r_1^3}{z^2} + \frac{2^{2/3}\Gamma(\frac{5}{3})}{108}\frac{r_1^5}{z^{8/3}} - \cdots\right\}, \tag{5.8}$$

which is also obtained by direct differentiation of (5.5). Figure 3 shows u_1 as a function of r_1 for some values of z.

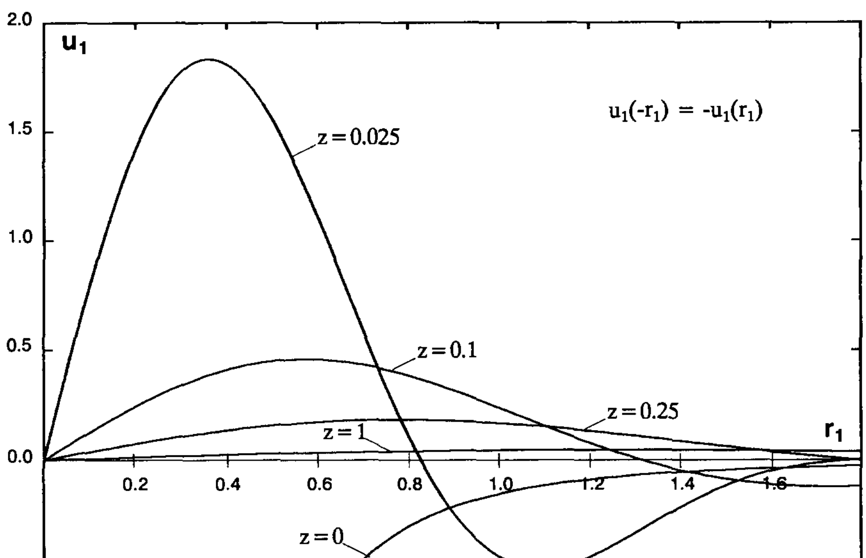

Fig. 3. The radial velocity u_1 as function of r_1 for some values of z.

For $r_1 > 0$ the axial velocity w_1 is equal to

$$w_1 = -\frac{\partial \psi_1}{\partial r_1} = \frac{3z}{r_1^4}\frac{d\psi_1}{d\tau} = \frac{3z}{4\pi r_1^4}\int_0^\infty y^2 \sin y\, e^{-y^3\tau/2}\, dy\,. \tag{5.9}$$

For $z = 0$ this gives $w_1 = 0$, while for z finite and $r_1 \to \infty$ $(\tau \to 0)$ we obtain

$$w_1 = -\frac{3z}{2\pi r_1^4}\,. \tag{5.10}$$

w_1 is an even function of r_1, so $w(-r_1) = w(r_1)$.

Finally, by differentiation of (4.7) we have for $z = 0$, $w_1 = \frac{1}{2}\delta(r_1)$, where $\delta(x)$ is the Dirac δ-function.

The expansion of w_1 for $\tau \to \infty$ becomes

$$w_1 = \frac{1}{2\pi}\left\{\frac{2^{1/3}\Gamma(\frac{1}{3})}{3z^{1/3}} - \frac{r_1^2}{3z} + \frac{2^{2/3}\Gamma(\frac{5}{3})}{36}\frac{r_1^4}{z^{5/3}} - \cdots\right\}. \tag{5.11}$$

Figure 4 shows w_1 as a function of r_1 for some values of z.

For calculating the remaining variables v_1 and p_1 we proceed as follows. In order to differentiate w_1 to r_1, we replace y by ωr_1 in (5.9)

$$w_1 = \frac{3z}{4\pi r_1}\int_0^\infty \omega^2 \sin \omega r_1\, e^{-\omega^3 z/2}\, d\omega\,.$$

By using

$$\frac{d\, e^{-\omega^3 z/2}}{d\omega} = -\frac{3}{2}\omega^2 z\, e^{-\omega^3 z/2}$$

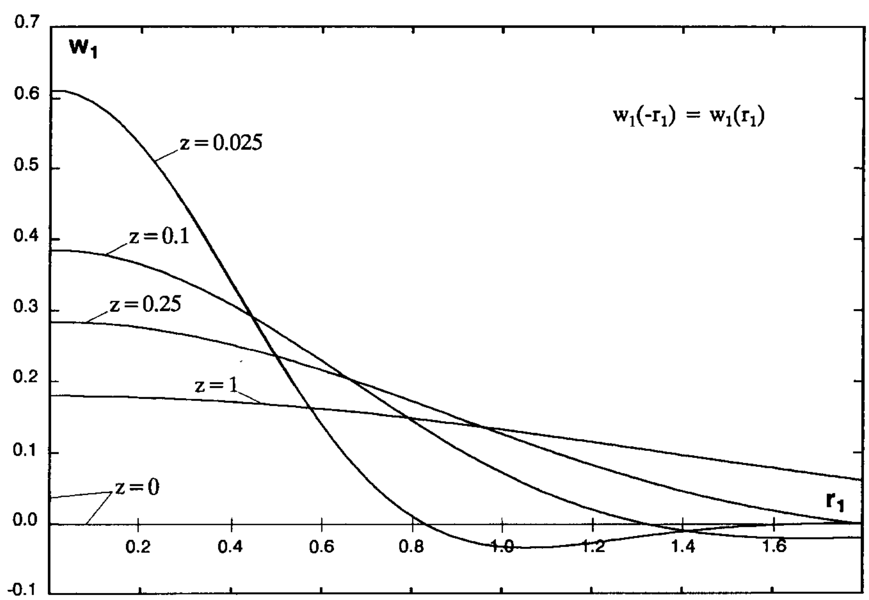

Fig. 4. The axial velocity w_1 as function of r_1 for some values of z.

and applying partial integration the result for w_1 becomes

$$w_1 = \frac{1}{2\pi} \int_0^\infty \cos \omega r_1 \, e^{-\omega^3 z/2} \, d\omega \,. \tag{5.12}$$

Then, from (4.3) we obtain

$$\frac{\partial p_1}{\partial z} = \frac{\partial^2 w_1}{\partial r_1^2} = -\frac{1}{2\pi} \int_0^\infty \omega^2 \cos \omega r_1 \, e^{-\omega^3 z/2} \, d\omega \,. \tag{5.13}$$

Since direct integration to z produces a divergent integral, we first calculate

$$\frac{\partial^2 p_1}{\partial r_1 \, \partial z} = \frac{1}{2\pi} \int_0^\infty \omega^3 \sin \omega r_1 \, e^{-\omega^3 z/2} \, d\omega$$

and then integrate to z. Hence

$$\frac{\partial p_1}{\partial r_1} = -\frac{1}{\pi} \int_0^\infty \sin \omega r_1 \{ e^{-\omega^3 z/2} + C(r_1) \} \, d\omega \,. \tag{5.14}$$

Again from (4.3)

$$v_1 = -\frac{1}{2\pi} \int_0^\infty \sin \omega r_1 \{ e^{-\omega^3 z/2} + C(r_1) \} \, d\omega \,.$$

Furthermore, we have $2u_1 = \partial^2 v_1 / \partial r_1^2$ and thus

$$u_1 = \frac{1}{4\pi} \int_0^\infty \omega^2 \sin \omega r_1 \, e^{-\omega^3 z/2} \, d\omega + \frac{1}{4\pi} \int_0^\infty \sin \omega r_1 \, e^{-\omega^3 z/2} \left\{ \omega^2 C(r_1) - \frac{d^2 C_1}{dr_1^2} \right\} d\omega \,.$$

Comparing this result with (5.6) we see that the second term must vanish. This leads to $C(r_1) = 0$ since the possibilities $C(r_1) = \sinh wr_1$ or $\cosh wr_1$ are excluded as $v_1 \to 0$ for $r_1 \to \infty$. Hence

$$v_1 = -\frac{1}{2\pi r_1} \int_0^\infty \sin y \, e^{-y^3 \tau/2} \, dy \,. \tag{5.15}$$

Writing (5.12) also in terms of y, it follows that w_1 and v_1 are the real and imaginary parts of

$$\frac{1}{2\pi r_1} \int_0^\infty e^{-iy} \, e^{-y^3 \tau/2} \, dy \,,$$

which is in agreement with [3].

Finally, (5.15) shows that for $\tau \downarrow 0$ ($z \downarrow 0$, $r_1 > 0$ or z finite and $r_1 \to \infty$) we have

$$v_1 = -\frac{1}{2\pi r_1} \,. \tag{5.16}$$

The expansion of v_1 for $\tau \to \infty$ is

$$v_1 = -\frac{1}{3\pi} \left\{ \frac{\Gamma(\frac{2}{3})}{2^{1/3}} \frac{r_1}{z^{2/3}} - \frac{2^{1/3}\Gamma(\frac{4}{3})}{6} \frac{r_1^3}{z^{4/3}} + \frac{1}{60} \frac{r_1^5}{z^2} - \cdots \right\} . \tag{5.17}$$

Figure 5 shows v_1 as a function of r_1 for some values of z. v_1 is an odd function of r_1, $v_1(-r_1) = -v_1(r_1)$.

It follows also from (5.13) that

$$\frac{\partial p_1}{\partial z} = -\frac{1}{2\pi r_1^3} \int_0^\infty y^2 \cos y \, e^{-y^3\tau/2} \, dy .$$

Expanding this again for $\tau \to \infty$ the result is

$$\frac{\partial p_1}{\partial z} = -\frac{1}{3\pi} \left\{ \frac{1}{z} - \frac{\Gamma(\frac{5}{3})}{2^{1/3}} \frac{r_1^2}{z^{5/3}} + \frac{2^{1/3}\Gamma(\frac{7}{3})}{12} \frac{r_1^4}{z^{7/3}} - \cdots \right\},$$

which agrees with differentiation of (5.11).

Integrating, we obtain as expansion of p_1 near $r_1 = 0$

$$p_1 = -\frac{1}{3\pi} \left\{ \ln z + \frac{3\Gamma(\frac{5}{3})}{2^{4/3}} \frac{r_1^2}{z^{2/3}} - \frac{2^{1/3}\Gamma(\frac{7}{3})}{16} \frac{r_1^4}{z^{4/3}} + \cdots \right\} + C_1(r_1) .$$

For arbitrary values of r_1 and $z > 0$, we can write

$$p_1 = -\frac{1}{3\pi} \ln z + \int_0^{r_1} \frac{\partial p_1}{\partial r_1} \, dr_1 + C_1(0) .$$

Since the pressure is only fixed up to an arbitrary constant, we may take $C_1(0) = 0$. Substituting the value for $\partial p_1/\partial r_1$ from (5.14) with $C(r_1) = 0$ and performing the integration to r_1, we find for positive values of r_1

$$p_1 = -\frac{1}{3\pi} \ln z - \frac{1}{\pi} \int_0^\infty \frac{1 - \cos y}{y} e^{-y^3\tau/2} \, dy , \tag{5.18}$$

where again y has been written for ωr_1.

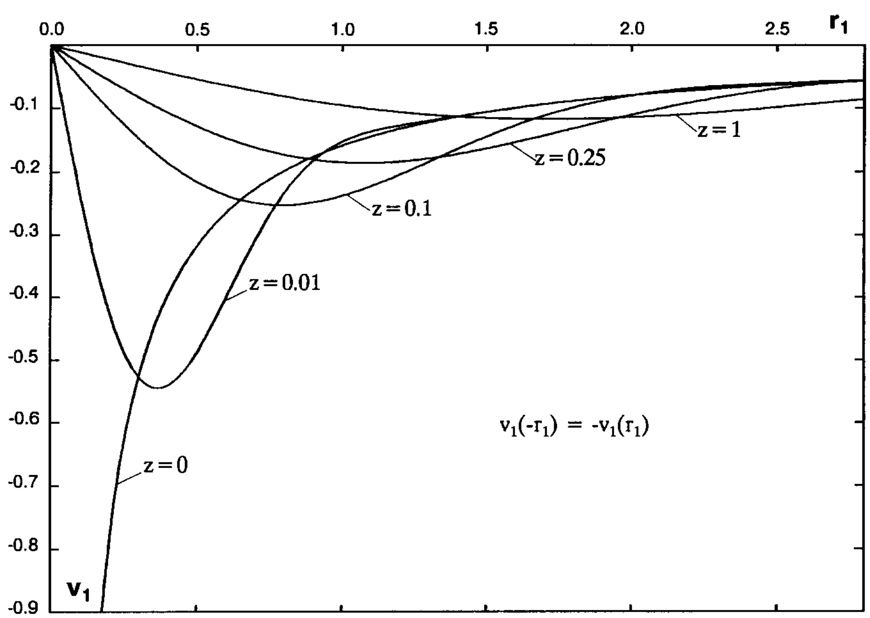

Fig. 5. The azimuthal velocity v_1 as function of r_1 for some values of z.

In order to investigate the behaviour of p_1 for finite z and $r_1 \to \infty$, we have to evaluate the integral in (5.18) for small values of τ. This integral is written as

$$\lim_{a\to 0}\left\{\int_a^\infty \frac{1}{y}\, e^{-y^3\tau/2}\, dy - \int_a^\infty \frac{\cos y}{y}\, e^{-y^3\tau/2}\, dy\right\}, \quad a>0\,. \tag{5.19}$$

The first integral is

$$\int_a^\infty \frac{1}{y}\, e^{-y^3\tau/2}\, dy = \frac{1}{3}\int_{\frac{1}{2}a^3\tau}^\infty \frac{e^{-u}}{u}\, du = \frac{1}{3}\, E_1\!\left(\frac{1}{2}\, a^3\tau\right),$$

according to [6], formula (5.1.1). For small values of a we have, see [6], (5.1.11)

$$\frac{1}{3}\, E_1\!\left(\frac{1}{2}\, a^3\tau\right) = \frac{1}{3}\left\{-\gamma - \ln\frac{1}{2}\, a^3\tau + O(a^3\tau)\right\},$$

where γ is Euler's constant. Hence in the limit $a \to 0$

$$\int_a^\infty \frac{1}{y}\, e^{-y^3\tau/2}\, dy = -\frac{1}{3}\,\gamma + \frac{1}{3}\ln 2 - \ln a - \frac{1}{3}\ln\tau + O(a^3\tau)\,.$$

For small values of τ the second integral in (5.19) is reduced as follows

$$\lim_{a\to 0}\int_a^\infty \frac{\cos y}{y}\, e^{-y^3\tau/2}\, dy = \lim_{a\to 0}\int_a^\infty \frac{\cos y}{y}\, dy - 15\tau^2 + O(\tau^4)\,,$$

where the exponential has been expanded. From [6], formulae (5.2.27) and (5.2.16) we have for $a \to 0$

$$\int_a^\infty \frac{\cos y}{y}\, dy = -\mathrm{Ci}(a) = -\gamma - \ln a\,.$$

The conclusion is that

$$\int_0^\infty \frac{1-\cos y}{y}\, e^{-y^3\tau/2}\, dy = \frac{2}{3}\,\gamma + \frac{1}{3}\ln 2 - \frac{1}{3}\ln\tau + 15\tau^2$$

and, substituting in (5.18), for $\tau \to 0$ the pressure becomes equal to

$$p_1 = -\frac{1}{\pi}\ln r_1 - \frac{1}{3\pi}(2\gamma + \ln 2) - \frac{15z^2}{\pi r_1^6}\,.$$

Since the pressure is an even function of r_1 we can write for $|\tau| \to 0$

$$p_1 = -\frac{1}{\pi}\ln|r_1| - 0.1960341 - \frac{15z^2}{\pi r_1^6}\,. \tag{5.20}$$

For arbitrary values of r_1 and $z>0$ we have (5.18). For some values of z, p_1 is given as function of r_1 in Fig. 6. It may be remarked that (5.10) and (5.20) satisfy the relation $\partial p_1/\partial z = \partial^2 w_1/\partial r_1^2$ in (4.3).

The infinitely large pressure at the singularity $r_1 = 0$, $z_1 = 0$ is the fundamental reason of the deviation of the flow from the Ekman boundary layer toward the Stewartson layer.

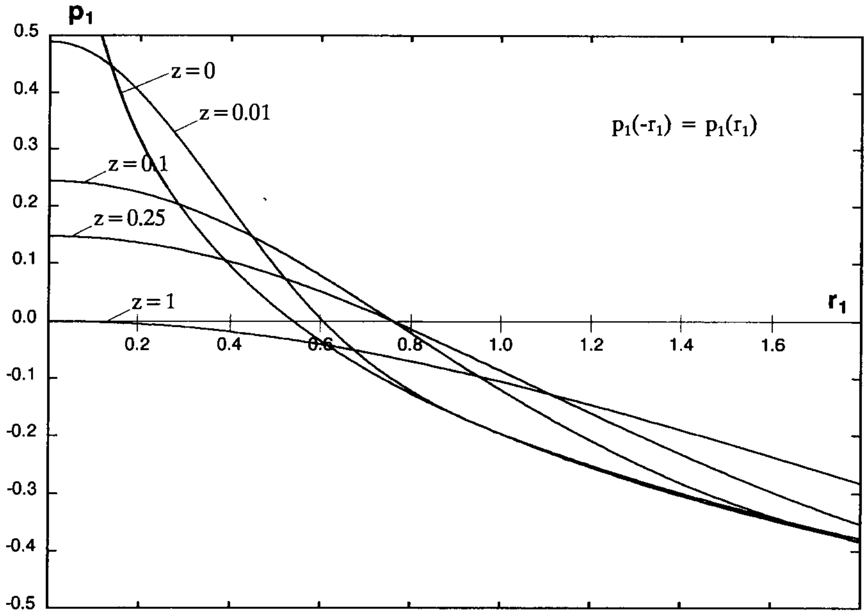

Fig. 6. The pressure p_1 as function of r_1 for some values of z.

6. The inner, outer and upper regions

The flow in the inner region for $r\uparrow 1$ should be matched to the flow in the Stewartson layer for $r_1 \to -\infty$. With $r_1 = -E^{-1/3}(1-r)$ and taking into account the even or odd character of the functions, we find from (4.2), (5.7), (5.10), (5.16) and (5.20), for $r\uparrow 1$

$$\left.\begin{aligned}
\psi &= \frac{1}{2}\,\varepsilon E^{1/2}\,,\\
u &= \varepsilon E^{3/2}\,\frac{1}{2\pi(1-r)^3}\,,\\
v &= (1-\varepsilon)r + \varepsilon E^{1/2}\,\frac{1}{2\pi(1-r)}\,,\\
w &= -\varepsilon E^{1/2} - \varepsilon E^{3/2}\,\frac{3z}{2\pi(1-r)^4}\,,\\
p &= \frac{1}{2}\,(1-\varepsilon)^2 r^2 + \frac{1}{3\pi}\,\varepsilon E^{1/2}\ln E - \varepsilon E^{1/2}\left\{\frac{1}{\pi}\ln(1-r) + 0.1960341\right\}.
\end{aligned}\right\} \qquad (6.1)$$

In general, we can write in the inner region

$$\left.\begin{aligned}
\psi &= \frac{1}{2}\,\varepsilon E^{1/2} r^2 + \varepsilon E^{3/2}\psi_i\,,\\
u &= \varepsilon E^{3/2} u_i\,,\\
v &= (1-\varepsilon)r + \varepsilon E^{1/2} v_i\,,\\
w &= -\varepsilon E^{1/2} + \varepsilon E^{3/2} w_i\,,\\
p &= \frac{1}{2}\,(1-\varepsilon)^2 r^2 + \frac{1}{3\pi}\,\varepsilon E^{1/2}\ln E + \varepsilon E^{1/2} p_i\,,
\end{aligned}\right\} \qquad (6.2)$$

where the limiting-values of the variables u_i etc. for $r \uparrow 1$ are given by (6.1) while

$$\lim_{r \uparrow 1} \psi_i = \frac{z}{2\pi(1-r)^3} .$$

The equations to be satisfied by the variables are

$$\left.\begin{aligned}
&2v_i = \frac{\partial p_i}{\partial r} , \\
&2u_i = \frac{\partial^2 v_i}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_i}{r}\right) , \\
&\frac{\partial p_i}{\partial r} = 0 , \\
&u_i = \frac{1}{r}\frac{\partial \psi_i}{\partial z} , \qquad w_i = -\frac{1}{r}\frac{\partial \psi_i}{\partial r} .
\end{aligned}\right\} \tag{6.3}$$

Since p_i is independent of z, it follows from the first equation, that also v_i is independent of z which is the reason that the term $\partial^2 v_i/\partial z^2$ could be omitted in the second equation. Only ψ_i and w_i are linear in z, the other variables being independent of z.

In the outer region we have

$$\begin{aligned}
&\psi = \varepsilon E^{3/2} \psi_0 , \\
&u = \varepsilon E^{3/2} u_0 , \\
&v = (1-\varepsilon) r + \varepsilon E^{1/2} v_0 , \\
&w = \varepsilon E^{3/2} w_0 , \\
&p = \frac{1}{2}(1-\varepsilon)^2 r^2 + \frac{1}{3\pi} \varepsilon E^{1/2} \ln E + \varepsilon E^{1/2} p_0 .
\end{aligned}$$

The limiting values for $r \downarrow 1$ are

$$\begin{aligned}
&\psi_0 \to -\frac{z}{2\pi(r-1)^3} , \qquad u_0 \to -\frac{1}{2\pi(r-1)^3} , \qquad v_0 \to -\frac{1}{2\pi(r-1)} , \\
&w_0 \to -\frac{3z}{2\pi(r-1)^4} , \qquad p_0 \to -\left\{\frac{1}{\pi}\ln(r-1) + 0.1960341\right\} .
\end{aligned} \tag{6.4}$$

The variables in the outer region also satisfy (6.3). Again ψ_0 and z_0 are linear in z, the other variables are independent of z.

The equations (6.3) are only modified when z becomes $O(E^{-1})$. We then have for the upper region

$$\psi = \varepsilon E^{1/2} \psi_u , \qquad u = \varepsilon E^{3/2} u_u , \qquad v = (1-\varepsilon) r + \varepsilon E^{1/2} v_u ,$$

$$w = \varepsilon E^{1/2} w_u , \qquad p = \frac{1}{2}(1-\varepsilon)^2 r^2 + \frac{1}{3\pi} \varepsilon E^{1/2} \ln E + \varepsilon E^{1/2} p_u .$$

With $z = E^{-1} z_u$ the equations become

$$2v_u = \frac{\partial p_u}{\partial r} ,$$

$$2u_u = \frac{\partial^2 v_u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v_u}{r}\right),$$

$$\frac{\partial p_u}{\partial z_u} = \frac{\partial^2 w_u}{\partial r^2} + \frac{1}{r}\frac{\partial w_u}{\partial r},$$

$$u_u = \frac{1}{r}\frac{\partial \psi_u}{\partial z_u}, \qquad w_u = -\frac{1}{r}\frac{\partial \psi_u}{\partial r}.$$

Only the third equation is changed in comparison with (6.3). For $z_u \downarrow 0$ the limits of ψ_u and w_u are different in case r is smaller or larger than 1

$$r<1: \quad \psi_u \to \frac{1}{2} r^2, \quad w_u \to -1$$

$$r>1: \quad \psi_u \to 0, \quad w_u \to 0.$$

It follows that the solution in the upper region contains a singularity at the point $r=1$, $z_u=0$. At the scale of the upper region, the Stewartson layer is reduced to the point $r=1$, $z_u=0$. In the upper region it no longer exists as a layer but its influence in the whole region is apparent through the singularity. The variable $\tau = z/r_1^3$ of the Stewartson layer becomes $z_u/(r-1)^3$ in the upper region.

7. Second approximation in the Stewartson layer

It was shown in Section 6 that the Stewartson layer gives rise to axial velocities $O(E^{3/2})$ in the inner and outer regions. However, in the inner region there is a more important axial velocity $w = -\varepsilon E^{1/2}$, due to the Ekman layer, which is lacking in the outer region. The second approximation in the Stewartson layer will show how the transition from $w = -\varepsilon E^{1/2}$ to $w = o(E^{1/2})$ occurs. The fundamental equation is given by (4.6) as

$$\frac{\partial^6 \psi_2}{\partial r_1^6} + 4\frac{\partial^2 \psi_2}{\partial z^2} = 3\frac{\partial^5 \psi_1}{\partial r_1^5}. \tag{7.1}$$

Boundary conditions are

$z=0$: $\quad \psi_2 = r_1 U(-r_1)$,

$z \to \infty$: $\quad \psi_2$ is bounded,

$r_1 \to -\infty$: $\quad \psi_2 \to r_1$,

$r_1 \to \infty$: $\quad \psi_2 \to 0$.

The solution of ψ_2 is again found by aid of Fourier transformation

$$F_2(\omega, z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi_2(r_1, z)\, e^{i\omega r_1}\, dr_1, \quad \text{Im}\,\omega < 0$$

The transformed equation is

$$4\frac{d^2F_2}{dz_2} - \omega^6 F_2 = -3i\omega^5 F_1 .$$

Substituting F_1 from Section 5 we obtain

$$4\frac{d^2F_2}{dz^2} - \omega^6 F_2 = -\frac{3\omega^4}{2\sqrt{2\pi}}\, e^{-|\omega|^3 z/2} .$$

Due to the boundedness of F_2 for $z \to \infty$, the solution can be written in the form

$$F_2 = A\, e^{-|\omega|^3 z/2} + Bz\, e^{-|\omega|^3 z/2} .$$

Fourier transformation of the boundary condition for $z = 0$ yields

$$F_2(\omega, 0) = A = \frac{1}{\omega^2\sqrt{2\pi}} ,$$

while substitution of F_2 in the differential equation gives

$$B = \frac{3|\omega|}{8\sqrt{2\pi}} .$$

The solution for ψ_2 then becomes

$$\begin{aligned}\psi_2(r_1, z) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\omega^2} + \frac{3|\omega|z}{8}\right) e^{-i\omega r_1}\, e^{-|\omega|^3 z/2}\, d\omega \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\frac{1}{\omega^2}\, e^{-i\omega r_1}(e^{-|\omega|^3 z/2} - 1)\, d\omega + \int_{-\infty}^{\infty}\frac{3|\omega|z}{8}\, e^{-i\omega r_1}\, e^{-|\omega|^3 z/2}\, d\omega \right. \\ &\quad \left. + \int_{-\infty}^{\infty}\frac{1}{\omega^2}\, e^{-i\omega r_1}\, d\omega\right] .\end{aligned}$$

The last integral has a double pole at $z = 0$ with residue $-ir_1$. For $r_1 < 0$ the integration path is closed by the infinitely large semi-circle in the half plane Im $\omega > 0$ and by the semi-circle in the half plane Im $\omega < 0$ if $r_1 > 0$. The result is $2\pi r_1 U(-r_1)$. In the other integrals we replace $e^{-i\omega r_1}$ by $\cos\omega r_1 - i\sin\omega r_1$. Remarking that the integrals with $\cos\omega r_1$ are even in ω but those with $\sin\omega r_1$ odd, the result is

$$\psi_2(r_1, z) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos\omega r_1}{\omega^2}\,(e^{-\omega^3 z/2} - 1)\, d\omega + \frac{3z}{8\pi}\int_0^{\infty}\omega\cos\omega r_1\, e^{-\omega^3 z/2}\, d\omega + r_1 U(-r_1) . \tag{7.2}$$

Expanding the exponential we find for $z \downarrow 0$

$$\psi_2(r_1, z) = r_1 U(-r_1) + \frac{z}{8\pi r_1^2} + O\left(\frac{z^3}{r_1^8}\right) .$$

Except for the first term, $\psi_2(r_1, z)$ is even in r_1. The axial velocity will be calculated from the formula $w_2 = -\partial\psi_2/\partial r_1 + r_1\,\partial\psi_1/\partial r_1$, see (4.5).

From (7.2) and (5.2) we have

$$\frac{\partial \psi_2}{\partial r_1} = -\frac{1}{\pi}\int_0^\infty \frac{\sin \omega r_1}{\omega}(\mathrm{e}^{-\omega^3 z/2} - 1)\,\mathrm{d}\omega - \frac{3z}{8\pi}\int_0^\infty \omega^2 \sin \omega r_1\, \mathrm{e}^{-\omega^3 z/2}\,\mathrm{d}\omega + U(-r_1)\,,$$

$$\frac{\partial \psi_1}{\partial r_1} = \frac{1}{2\pi}\int_0^\infty \cos \omega r_1 (1 - \mathrm{e}^{-\omega^3 z/2})\,\mathrm{d}\omega - \frac{1}{2}\,\delta(r_1)\,.$$

Using

$$\int_0^\infty \frac{\sin \omega r_1}{\omega}\,\mathrm{d}\omega = \frac{\pi}{2} - \pi U(-r_1)$$

and

$$\int_0^\infty \cos \omega r_1\,\mathrm{d}\omega = \pi\delta(r_1)\,,$$

we obtain

$$w_2 = \frac{1}{2\pi}\int_0^\infty \left(2\,\frac{\sin \omega r_1}{\omega} - r_1 \cos \omega r_1\right)\mathrm{e}^{-\omega^3 z/2}\,\mathrm{d}\omega + \frac{3z}{8\pi}\int_0^\infty \omega^2 \sin \omega r_1\, \mathrm{e}^{-\omega^3 z/2}\,\mathrm{d}\omega - \frac{1}{2}\,.$$

Partial integration of the last integral gives

$$w_2 = \frac{1}{4\pi}\int_0^\infty \left(4\,\frac{\sin \omega r_1}{\omega} - r_1 \cos \omega r_1\right)\mathrm{e}^{-\omega^3 z/2}\,\mathrm{d}\omega - \frac{1}{2}$$

or

$$w_2 = \frac{1}{4\pi}\int_0^\infty \left(4\,\frac{\sin y}{y} - \cos y\right)\mathrm{e}^{-y^3\tau/2}\,\mathrm{d}y - \frac{1}{2} \quad \text{if } \tau = \frac{z}{r_1^3} > 0$$

and

$$w_2 = -\frac{1}{4\pi}\int_0^\infty \left(4\,\frac{\sin y}{y} - \cos y\right)\mathrm{e}^{y^3\tau/2}\,\mathrm{d}y - \frac{1}{2} \quad \text{if } \tau = \frac{z}{r_1^3} < 0\,.$$

It is seen that w_2 only depends on τ. For $\tau\downarrow 0$ we have

$$w_2 = \frac{7\tau}{4\pi} + O(\tau^3)\,,$$

while for $\tau < 0$

$$w_2(\tau) = -1 - w_2(-\tau) = -1 + \frac{7\tau}{4\pi} + O(\tau^3)$$

holds. Results for w_2 are presented in Table 2 and Fig. 7.

The contribution to w for $|r_1| \to \infty$ is

$$\varepsilon E^{1/2}\left\{-U(-r_1) + \frac{7z}{4\pi r_1^3}\right\}.$$

Table 2. The function $w_2(\tau)$, $\tau = 0.027171$ gives maximum of $w_2(\tau)$

τ	$w_2(\tau)$
0	0
0.000 01	0.000 005 570
0.027 171	0.041 872 659
0.075 277	0
0.1	−0.023 631 987
0.219 096	−0.1
0.617 158	−0.2
2.281 694	−0.3
19.107 978	−0.4
∞	−0.5

Matching to the inner region gives for $r \uparrow 1$

$$w = -\varepsilon E^{1/2} - \varepsilon E^{3/2} \frac{7z}{4\pi(1-r)^3}$$

and the outer region

$$w = \varepsilon E^{3/2} \frac{7z}{4\pi(r-1)^3} .$$

This yields further terms in the expansions for $r \to 1$ of the solutions in the inner and outer regions as given by (6.1) and (6.4).

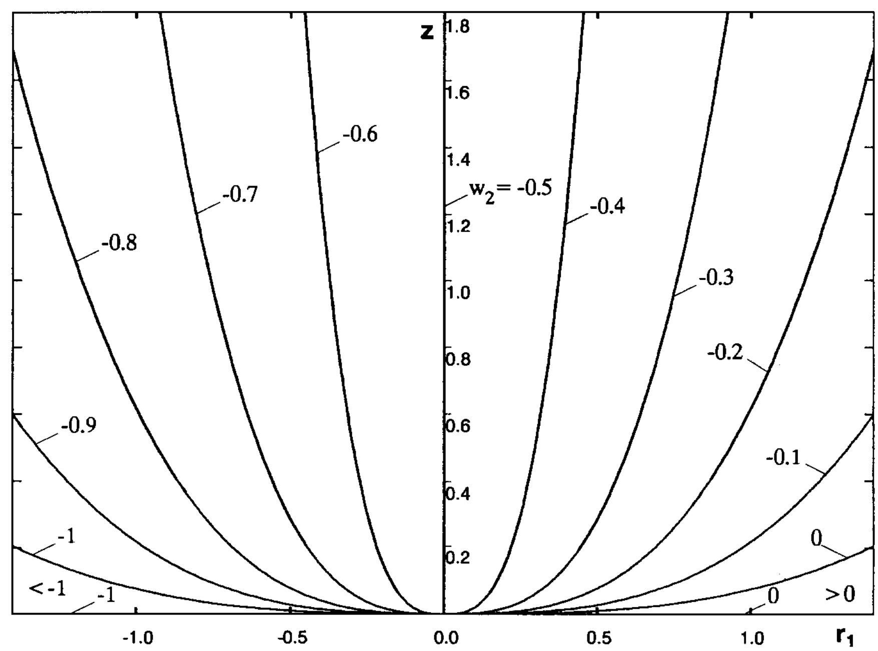

Fig. 7. The axial velocity w_2 in the $r_1 - z$ plane.

The radial velocity follows from $u_2 = \partial\psi_2/\partial z - r_1\,\partial\psi_1/\partial z$. Using (7.2) and (5.6) the result turns out to be

$$u_2 = -\frac{1}{8\pi}\int_0^\infty (3\omega\cos\omega r_1 + \omega^2 r_1 \sin\omega r_1)\, e^{-\omega^3 z/2}\, d\omega$$

or

$$u_2 = -\frac{1}{8\pi r_1^2}\int_0^\infty (3y\cos y + y^2\sin y)\, e^{-y^3\tau/2}\, dy \quad \text{if } \tau > 0\,.$$

u_2 is an even function of r_1. The contribution to u for $|r_1| \to \infty$ is

$$\varepsilon E^{5/6}\,\frac{5}{8\pi r_1^2}\,,$$

which gives by matching to the inner and outer regions a term

$$\varepsilon E^{3/2}\,\frac{5}{8\pi (r-1)^2}$$

in addition to the terms already obtained in (6.1) and (6.4).

After elaborate calculations along the same lines as in Section 5 we find the following results for p_2 and v_2

$$p_2 = -\frac{r_1}{2\pi}\left\{\ln z + \int_0^\infty \frac{y(3+\cos y) - 4\sin y}{y^2}\, e^{-y^3\tau/2}\, dy + C_3\right\} \quad \text{if } \tau > 0\,,$$

$$v_2 = -\frac{1}{4\pi}\left\{\ln z + \int_0^\infty \frac{3(1-\cos y) - y\sin y}{y}\, e^{-y^3\tau/2}\, dy + C_3\right\} \quad \text{if } \tau > 0\,.$$

p_2 is odd and v_2 is even in r_1. For small numbers of τ we obtain

$$p_2 = \frac{r_1}{2\pi}(-3\ln r_1 + 27\tau^2 + C_4)\,, \quad \tau\downarrow 0\,,$$

$$v_2 = \frac{1}{4\pi}(-3\ln r_1 - 135\tau^2 + C_4 - 3)\,, \quad \tau\downarrow 0\,,$$

where $C_4 = 4 - 2\gamma - \ln 2 - C_3$. The constant C_3 is determined by the flow outside the Stewartson layer.

Matching of p_2 and v_2 gives further terms in the asymptotic expansions of p_i, p_0, v_i and v_0 for $r \to 1$ in the inner and outer regions.

8. Conclusions

A rotating disk placed in a fluid rotating coaxially with a slightly different angular velocity shows at its edge a Stewartson layer of width $O(E^{1/3})$ and height $O(E^{-1})$ provided $|\varepsilon| < O(E^{1/6})$. This layer is due to the sudden deflection of the boundary layer flow into an axial flow if $\varepsilon > 0$ and reversely if $\varepsilon < 0$. The deflection is caused by a logarithmic pressure singularity at $r_1 = 0$, $z = 0$.

Velocity and pressure distributions in the Stewartson layer have been evaluated and calculated for some values of z as functions of r_1, see Figs 2 to 6. Due to the occurrence of the similarity parameter τ they have a simple analytic form.

The orders of magnitude of the various quantities in the Stewartson layer are

stream function	ψ	$O(E^{1/2})$,
radial velocity	u	$O(E^{1/2})$,
azimuthal velocity	v	$O(E^{1/6})$,
axial velocity	w	$O(E^{1/6})$,
pressure	p	$O(E^{1/2})$.

The everywhere present azimuthal velocity $v = (1 - \varepsilon)r$ and corresponding pressure $p = \frac{1}{2}(1 - \varepsilon)^2 r^2$ have been left out of account. The orders of magnitude in the inner and outer regions are respectively

	ψ	u	v	w	p
inner	$O(E^{1/2})$	$O(E^{3/2})$	$O(E^{1/2})$	$O(E^{1/2})$	$O(E^{1/2}\ln E) + O(E^{1/2})$,
outer	$O(E^{3/2})$	$O(E^{3/2})$	$O(E^{1/2})$	$O(E^{3/2})$	$O(E^{1/2}\ln E) + O(E^{1/2})$.

The reduction of the axial velocity of $O(E^{1/2})$ in the inner region to $O(E^{3/2})$ in the outer region has been investigated with the aid of the second approximation of the solution of the differential equations, see Section 5. This approximation, which is valid for $|\varepsilon| < O(E^{1/2})$ gives contributions in the Stewartson layer of the following orders of magnitude

ψ	u	v	w	p
$O(E^{5/6})$	$O(E^{5/6})$	$O(E^{1/2})$	$O(E^{1/2})$	$O(E^{5/6})$.

In the upper region the orders of magnitude are the same as in the inner region, which means that w is also $O(E^{1/2})$.

Note added in proof

It appears that the homogeneous differential equation

$$\frac{\partial^6 \psi}{\partial r_1^6} + 4\,\frac{\partial^2 \psi}{\partial z^2} = 0$$

has additional solutions. These might be excited by the Ekman layer at the singular point $r_1 = 0$, $z = 0$. It means that the solution ψ_2 might be modified by an additional term containing an unknown factor. Whether this is the case should follow from an investigation of the region connecting the Stewartson and Ekman layers. Such investigation is being performed.

Acknowledgement

The author wishes to thank Kees Visser for programming the figures for this paper.

References

1. K. Stewartson, On almost rigid rotations. *J. Fluid Mech.* 3 (1957) 17–26.
2. H.P. Greenspan, *The Theory of Rotating Fluids*. Cambridge University Press, Cambridge (1968).
3. D.W. Moore and P.G. Saffman, The structure of free vertical shear layers in a rotating fluid and the motion produced by a slowly rising body. *Phil. Trans. Roy. Soc.* A264 (1969) 597–634.
4. R. Hide and C.W. Titman, Detached shear layers in a rotating fluid. *J. Fluid Mech.* 29 (1967) 39–60.
5. A.I. van de Vooren and E.F.F. Botta, Fluid flow induced by a rotating disk of finite radius. *J. Eng. Math.* 24 (1990) 55–71.
6. M. Abramowitz and J.A. Stegun, *Handbook of Mathematical Functions*. Dover (1965).